

COMPLETELY \mathfrak{m} -FULL IDEALS AND COMPONENTWISE LINEAR IDEALS

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ABSTRACT. We show that the class of completely \mathfrak{m} -full ideals coincides with the class of componentwise linear ideals in a polynomial ring over an infinite field.

1. INTRODUCTION

The notion of completely \mathfrak{m} -full ideals in a local ring was introduced by the second author [9], and the notion of componentwise linear ideals in a polynomial ring was introduced by Herzog and Hibi [5]. These ideals are two important classes of ideals having various interesting properties. In [6] the authors proved that these notion are equivalent in the class of graded ideals provided that their generic initial ideals with respect to the graded reverse lexicographic order are stable, and further conjectured that these notions are equivalent without adding the assumption on generic initial ideals. The purpose of this paper is to prove that the conjecture is true. The following is the main theorem.

Theorem 1.1. *Let K be an infinite field of any characteristic and I a graded ideal of the polynomial ring $R = K[x_1, \dots, x_n]$. Then I is a completely \mathfrak{m} -full ideal if and only if it is a componentwise linear ideal.*

The “if” part was proved in Proposition 18 of [6]. So we will show the “only if” part. For the proof of the “only if” part, we use the characterization theorem for componentwise linear ideals by Nagel and Römer [7]. Their result says that the following conditions are equivalent for a graded ideal I of $R = K[x_1, \dots, x_n]$.

- (i) I is a componentwise linear ideal.
- (ii) The generic initial ideal $\text{Gin}(I)$ of I is stable and $\mu(I) = \mu(\text{Gin}(I))$, where μ denotes the minimal number of generators of an ideal.

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In Section 5 we prove that if I is a completely \mathfrak{m} -full ideal then $\text{Gin}(I)$ is stable and $\mu(I) = \mu(\text{Gin}(I))$. In Section 2 we summarize basic notation and definitions. In Section 3 we introduce the notion of t -sequences for a graded ideal, and in Section 4 we give a characterization of completely \mathfrak{m} -full ideals in terms of the t -sequences. It plays a key role in the proof of Theorem 1.1. In section 6 we show that a theorem of Nagel-Römer ([7]) is an immediate consequence of Theorem 1.1.

2. NOTATION AND DEFINITIONS

Throughout this paper, we let K be an infinite field of any characteristic, $R = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K with the standard grading, and $\mathfrak{m} = (x_1, \dots, x_n)$ the graded maximal ideal. Let $\text{Gin}(I)$ denote the generic initial ideal of an ideal I of R with respect to the graded reverse lexicographic order induced by $x_1 > \dots > x_n$, and $H(M, j) = \dim_K M_j$ the Hilbert function of a graded module $M = \bigoplus_{j \geq 0} M_j$ over R . Let l and μ be the length and the minimal number of generators of a graded ideal I in R , respectively, hence $\mu(I) = l(I/\mathfrak{m}I)$. The type of a graded ideal I is the length of $(I : \mathfrak{m})/I$ as an ideal of R/I . It is equal to the last free rank in the minimal free resolution of R/I .

The definition of \mathfrak{m} -full ideal is due to Rees. We adapt the definition to graded ideals as follows.

Definition 2.1 ([8]). A graded ideal I of $R = K[x_1, \dots, x_n]$ is said to be \mathfrak{m} -full if there exists an element z of R such that $\mathfrak{m}I : z = I$.

Remark 2.2. Suppose that I is an \mathfrak{m} -full ideal of R . Then the equality $\mathfrak{m}I : z = I$ holds for a general linear form z of R ([8, Remark 2 (i)]).

We adapt the original definition of completely \mathfrak{m} -full ideals (defined in [9]) to the graded ideals as follows.

Definition 2.3 ([9]). Let I be a graded ideal of $R = K[x_1, \dots, x_n]$. We define the *completely \mathfrak{m} -full ideals* recursively as follows.

- (1) If $n = 0$ (i.e., if R is a field), then the zero ideal is completely \mathfrak{m} -full.
- (2) If $n > 0$, then I is completely \mathfrak{m} -full if $\mathfrak{m}I : z = I$ and $(I + zR)/zR$ is completely \mathfrak{m} -full as an ideal of R/zR , where z is a general linear form in R . (The definition makes sense by induction on n .)

Remark 2.4. For a completely \mathfrak{m} -full ideal I of $R = K[x_1, \dots, x_n]$, there exist n general linear forms z_n, z_{n-1}, \dots, z_1 in R satisfying the following conditions:

- (i) $\mathfrak{m}I : z_n = I$, i.e., I is \mathfrak{m} -full.
- (ii) $\overline{\mathfrak{m}I} : \overline{z_{n-i+1}} = \overline{I}$ in $\overline{R} = R/(I, z_n, \dots, z_{n-i+2})$ for all $i = 2, 3, \dots, n$, where $\overline{*}$ denotes the reduction mod $(I, z_n, \dots, z_{n-i+2})$.

In this case we say that $(I; z_n, z_{n-1}, \dots, z_1)$ has the complete \mathfrak{m} -full property.

Definition 2.5 ([5]). If I is a graded ideal of $R = K[x_1, \dots, x_n]$, then we write $I_{<j>}$ for the ideal generated by all homogeneous polynomials of degree j belong to I . We say that a graded ideal I of R is *componentwise linear* if $I_{<j>}$ has a linear resolution for all j .

Definition 2.6. A monomial ideal I of $R = K[x_1, \dots, x_n]$ is said to be *stable* if I satisfies the following condition: for each monomial $u \in I$, the monomial $x_i u / x_{m(u)}$ belongs to I for every $i < m(u)$, where $m(u)$ is the largest index j such that x_j divides u .

It is known that stable ideals are completely \mathfrak{m} -full ([10, Section 4] and [6, Example 17]), and also componentwise linear ([5, Example 1.1]).

3. THE t -SEQUENCE OF A GRADED IDEAL

The second author [10] defined the t -sequence for a completely \mathfrak{m} -full ideal. In this section we extend the notion of t -sequences to graded ideals in general. The following is a revised version of Theorem C in [8].

Theorem 3.1. *Let X_1, \dots, X_n be indeterminates over $R = K[x_1, \dots, x_n]$, let R' denote $S = R[X_1, \dots, X_n]$ localized at $\mathfrak{m}R[X_1, \dots, X_n]$ and set $Y = X_1x_1 + \dots + X_nx_n$ in R' . Let I be a graded ideal of R . Then we have the following.*

- (1) $l((IR' :_{R'} Y)/IR')$ is finite.
- (2) $l((IR' :_{R'} Y)/IR') \leq l((I :_R y)/I)$ for all linear forms y in R .
- (3) $l((IR' :_{R'} Y)/IR') = l((I :_R y)/I)$ for a general linear form y in R .

To prove this theorem we prepare a lemma.

Lemma 3.2. *Let I be a graded ideal of $R = K[x_1, \dots, x_n]$. Then $l((I : y)/I)$ is finite for general linear forms y of R .*

Proof. Let $\text{Ass}(I)$ be the set of associated prime ideals of I . If $\text{Ass}(I) = \{\mathfrak{m}\}$, then it is obvious that $l((I : y)/I)$ is finite for all linear forms y in R since I is \mathfrak{m} -primary. If $\mathfrak{m} \notin \text{Ass}(I)$ then $I : y = I$ for a general linear form y in R , because y is a non-zero divisor for R/I if y is general enough. Hence $l((I : y)/I) = 0$ in this case. So we assume that I is not \mathfrak{m} -primary and $\mathfrak{m} \in \text{Ass}(I)$. Let $I = \cap_{i=1}^u \mathfrak{q}_i$ be a minimal primary decomposition of I , where $\sqrt{\mathfrak{q}_1} = \mathfrak{m}$. Let y be a linear form of R such that $y \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(I) \setminus \{\mathfrak{m}\}$. It suffices to show that $l((I : y)/I)$ is finite. We have $I : y = \cap_{i=1}^u (\mathfrak{q}_i : y)$. Since $\sqrt{\mathfrak{q}_1 : y} = \mathfrak{m}$ and $\mathfrak{q}_i : y = \mathfrak{q}_i$ for $i > 1$, one sees that $(I : y)/I$ is annihilated by a power of \mathfrak{m} . This implies that $l((I : y)/I)$ is finite. \square

Proof of Theorem 3.1. By the first part of Theorem A in [8], we have the inequalities

$$(1) \quad l(R'/(I + \mathfrak{m}^{s+1})R' + YR') \leq l(R/(I + \mathfrak{m}^{s+1}) + yR)$$

for all $s \geq 0$. That is, the inequalities

$$\sum_{j=0}^s \mathrm{H}(R'/(IR' + YR'), j) \leq \sum_{j=0}^s \mathrm{H}(R/(I + yR), j)$$

hold for all $s \geq 0$. From the exact sequence

$$0 \rightarrow (I : y)/I \rightarrow R/I \xrightarrow{\times y} R/I \rightarrow R/(I + yR) \rightarrow 0,$$

it follows that

$$\mathrm{H}((I : y)/I, j) = \mathrm{H}(R/I, j) - \mathrm{H}(R/I, j+1) + \mathrm{H}(R/(I + yR), j+1)$$

for all $j \geq 0$. Similarly it follows that

$$\mathrm{H}((IR' : Y)/IR', j) = \mathrm{H}(R'/IR', j) - \mathrm{H}(R'/IR', j+1) + \mathrm{H}(R'/(IR' + YR'), j+1)$$

for all $j \geq 0$. Hence, since R/I and R'/IR' have the same Hilbert function, we obtain the inequalities

$$\sum_{j=0}^s \mathrm{H}((IR' : Y)/IR', j) \leq \sum_{j=0}^s \mathrm{H}((I : y)/I, j)$$

for all $s \geq 0$. Furthermore it follows by Lemma 3.2 that, for a general linear form y of R , the equalities

$$l((I : y)/I) = \sum_{j=0}^s \mathrm{H}((I : y)/I, j)$$

hold for all $s \gg 0$. Therefore we have

$$\sum_{j=0}^s \mathrm{H}((IR' : Y)/IR', j) \leq l((I : y)/I)$$

for all $s \gg 0$. Thus the assertions (1) and (2) are easily verified. The assertion (3) is also easy, since the equality in (1) holds for a general linear form y of R by the second part of Theorem A in [8]. \square

Definition 3.3. With the same notation as Theorem 3.1, we define $t(I)$ for a graded ideal I by

$$t(I) = l((IR' :_{R'} Y)/IR').$$

We call $t(I)$ the t -value of I . Note that the equality

$$t(I) = \min\{l((I : y)/I) \mid y \text{ is a linear form of } R\}$$

holds by Theorem 3.1.

Definition 3.4. Let $\{z_1, \dots, z_n\}$ be a set of generators of \mathfrak{m} consisting of general linear forms. Set

$$R^{(i)} = R/(z_{i+1}, z_{i+2}, \dots, z_n)R$$

for all $i = 0, 1, \dots, n-1$, and $R^{(n)} = R$. Let $t_i = t_i(I)$ denote the t -value of $IR^{(i+1)}$ for all $i = 0, 1, \dots, n-1$. Note that $t_0 = 1$. We call the sequence t_0, t_1, \dots, t_{n-1} the t -sequence of I . This is a generalization of the notion of t -sequences introduced by the second author in [10]. We will discuss it in Remark 3.6 below.

Remark 3.5. We show that the t -sequence of I is independent of a choice of general generators of \mathfrak{m} . We use the same notation as Definition 3.4. Let $\overline{z_i}$ be the image of z_i in $R^{(i)}$. From the exact sequence

$$0 \rightarrow (IR^{(i)} : \overline{z_i})/IR^{(i)} \rightarrow R^{(i)}/IR^{(i)} \xrightarrow{\times \overline{z_i}} R^{(i)}/IR^{(i)} \rightarrow R^{(i)}/(IR^{(i)} + \overline{z_i}R^{(i)}) \rightarrow 0,$$

it follows that

$$\begin{aligned} H((IR^{(i)} : \overline{z_i})/IR^{(i)}, j) &= H(R^{(i)}/(IR^{(i)} + \overline{z_i}R^{(i)}), j+1) \\ &\quad - H(R^{(i)}/IR^{(i)}, j+1) + H(R^{(i)}/IR^{(i)}, j) \end{aligned}$$

for all j . Set $R_{(i)} = R/(x_{i+1}, x_{i+2}, \dots, x_n)R$ for all $i = 0, 1, \dots, n-1$, $R_{(n)} = R$ and $J = \text{Gin}(I)$. Similarly we get

$$\begin{aligned} H((JR_{(i)} : \overline{x_i})/JR_{(i)}, j) &= H(R_{(i)}/(JR_{(i)} + \overline{x_i}R_{(i)}), j+1) \\ &\quad - H(R_{(i)}/JR_{(i)}, j+1) + H(R_{(i)}/JR_{(i)}, j) \end{aligned}$$

for all j , where $\overline{x_i}$ is the image of x_i in $R_{(i)}$. Hence, since

$$\begin{aligned} H(R^{(i)}/IR^{(i)}, j) &= H(R_{(i)}/JR_{(i)}, j) \\ \text{and } H(R^{(i)}/(IR^{(i)} + \overline{z_i}R^{(i)}), j) &= H(R_{(i)}/(JR_{(i)} + \overline{x_i}R_{(i)}), j) \end{aligned}$$

for general linear forms z_1, \dots, z_n by [1, Lemma 1.2], we have

$$l((IR^{(i)} : \overline{z_i})/IR^{(i)}) = l((JR_{(i)} : \overline{x_i})/JR_{(i)})$$

for all $i > 0$. This implies that the t -sequence of I is independent of a choice of general generators of \mathfrak{m} . This also implies that the t -sequence of I coincides with that of $\text{Gin}(I)$.

Remark 3.6. With the same notation as Definition 3.4, suppose that $(I; z_n, z_{n-1}, \dots, z_1)$ has the complete \mathfrak{m} -full property. Let t_0, t_1, \dots, t_{n-1} be the t -sequence of I . The definition of t -sequences given in [10, p. 238] implies that

$$t_i = \mu(IR^{(i+1)}) - \mu(IR^{(i)})$$

for all $i = 0, 1, \dots, n-1$. Here note that $\mu(IR^{(0)}) = 0$ because $R^{(0)} = K$ and $IR^{(0)} = 0$. Hence, since $(\mathfrak{m}I)R^{(i+1)} : \overline{z_{i+1}} = IR^{(i+1)}$, it follows from Lemma 4.3 in the next section that

$$t_i = l((IR^{(i+1)} : \overline{z_{i+1}})/IR^{(i+1)}).$$

This means that Definition 3.4 gives a generalization of the notion of t -sequences for completely \mathfrak{m} -full ideals.

4. A CHARACTERIZATION OF COMPLETELY \mathfrak{m} -FULL IDEALS

The purpose of this section is to prove the following.

Theorem 4.1. *Let I be a graded ideal of $R = K[x_1, \dots, x_n]$ and t_0, t_1, \dots, t_{n-1} the t -sequence of I . Set $B(I) = t_0 + t_1 + \dots + t_{n-1}$. Then the following conditions are equivalent.*

- (i) I is a completely \mathfrak{m} -full ideal.
- (ii) $\mu(I) = B(I)$.

We need a few lemmas for the proof of this theorem.

Lemma 4.2. *Let I be a graded ideal of R , z a linear form of R and \bar{I} the image of I in R/zR . Then*

- (1) $l((\mathfrak{m}I : z)/\mathfrak{m}I) = \mu(\bar{I}) + l((I : z)/I)$, and
- (2) $\mu(I) \leq \mu(\bar{I}) + l((I : z)/I)$.

Proof. From the exact sequence

$$0 \rightarrow (\mathfrak{m}I : z)/\mathfrak{m}I \rightarrow R/\mathfrak{m}I \xrightarrow{\times z} R/\mathfrak{m}I \rightarrow R/(\mathfrak{m}I + zR) \rightarrow 0,$$

it follows that

$$\begin{aligned} H(I/\mathfrak{m}I, j) &\leq H((\mathfrak{m}I : z)/\mathfrak{m}I, j) \\ &= H(R/\mathfrak{m}I, j) - H(R/\mathfrak{m}I, j+1) + H(R/(\mathfrak{m}I + zR), j+1) \\ &= H(R/\mathfrak{m}I, j) - H(R/\mathfrak{m}I, j+1) \\ &\quad + H(R/(I + zR), j+1) + H((I + zR)/(\mathfrak{m}I + zR), j+1) \end{aligned}$$

for all j . Similarly, from the exact sequence

$$(2) \quad 0 \rightarrow (I : z)/I \rightarrow R/I \xrightarrow{\times z} R/I \rightarrow R/(I + zR) \rightarrow 0,$$

it follows that

$$H(R/(I + zR), j+1) = H((I : z)/I, j) - H(R/I, j) + H(R/I, j+1)$$

for all j . Hence we see

$$\begin{aligned} H(I/\mathfrak{m}I, j) &\leq H((\mathfrak{m}I : z)/\mathfrak{m}I, j) \\ &= H(I/\mathfrak{m}I, j) - H(I/\mathfrak{m}I, j+1) \\ &\quad + H((I : z)/I, j) + H((I + zR)/(\mathfrak{m}I + zR), j+1) \end{aligned}$$

for all j . Thus we have

$$\begin{aligned} \mu(I) = l(I/\mathfrak{m}I) &\leq l((\mathfrak{m}I : z)/\mathfrak{m}I) \\ &= l((I + zR)/(\mathfrak{m}I + zR)) + l((I : z)/I) \\ &= \mu(\bar{I}) + l((I : z)/I). \end{aligned}$$

□

Lemma 4.3. *We use the same notation as Lemma 4.2. Then the following conditions are equivalent.*

- (i) $\mathfrak{m}I : z = I$.

$$(ii) \quad \mu(I) = \mu(\bar{I}) + l((I : z)/I).$$

Furthermore, if we assume that I is \mathfrak{m} -primary, these conditions are also equivalent to the following (iii).

$$(iii) \quad \mu(I) = \mu(\bar{I}) + l(R/(I + zR)).$$

Proof. (i) \Leftrightarrow (ii) is immediate from Lemma 4.2. (ii) \Leftrightarrow (iii) is obvious, because $l((I : z)/I) = l(R/(I + zR))$ from the exact sequence (2) above if I is \mathfrak{m} -primary. \square

Lemma 4.4. *Let I be a graded ideal of $R = K[x_1, \dots, x_n]$ and t_0, t_1, \dots, t_{n-1} the t -sequence of I . Set $B(I) = t_0 + t_1 + \dots + t_{n-1}$. Then*

$$\mu(I) \leq B(I).$$

Proof. We use induction on n . If $n = 1$, the equalities $\mu(I) = B(I) = 1$ hold. Let $n > 1$. By Lemma 4.2 (2), it follows that $\mu(I) \leq \mu(\bar{I}) + l((I : z)/I)$ for a general linear form z of R . Furthermore the inductive assumption implies that $\mu(\bar{I}) \leq B(\bar{I})$. Hence we have

$$\mu(I) \leq B(\bar{I}) + l((I : z)/I) = t_0 + t_1 + \dots + t_{n-1} = B(I),$$

as $B(\bar{I}) = t_0 + \dots + t_{n-2}$ and $l((I : z)/I) = t_{n-1}$. \square

Proof of Theorem 4.1. (i) \Rightarrow (ii) follows from Corollary 9 in [9]. (ii) \Rightarrow (i): We use induction on n . Let $n = 1$. Then the equalities $\mu(I) = B(I) = 1$ hold and any ideal of $K[x_1]$ is completely \mathfrak{m} -full. Let $n > 1$. Note that $B(I) = B(\bar{I}) + t_{n-1}$. Hence we have that $B(\bar{I}) \leq \mu(\bar{I})$, since $\mu(I) = B(I)$ and $\mu(I) \leq \mu(\bar{I}) + t_{n-1}$ by Lemma 4.2 (2).

On the other hand, the inequality $\mu(\bar{I}) \leq B(\bar{I})$ holds by Lemma 4.4, and hence the equality $B(\bar{I}) = \mu(\bar{I})$ holds. Therefore \bar{I} is completely \mathfrak{m} -full by the inductive assumption. Next we show that I is \mathfrak{m} -full. This follows from Lemma 4.3 and the equalities

$$\mu(I) = B(I) = B(\bar{I}) + t_{n-1} = \mu(\bar{I}) + l((I : z)/I)$$

for a general linear form z of R . \square

Corollary 4.5. *Let I be a graded ideal of $R = K[x_1, \dots, x_n]$, x a non-zero divisor mod I of degree one and \bar{I} the image of I in R/xR . Then I is a completely \mathfrak{m} -full ideal in R if and only if \bar{I} is a completely \mathfrak{m} -full ideal in R/xR .*

Proof. Since x is a non-zero divisor mod I of degree one, it follows that $\mu(\bar{I}) = \mu(I)$. Furthermore we have $B(\bar{I}) = B(I)$ by $l((I : x)/I) = 0$. Hence this follows from Theorem 4.1. \square

5. PROOF OF MAIN THEOREM 1.1

The following is a remark on a minimal generating set of an \mathfrak{m} -full ideal.

Remark 5.1. Suppose that I is an \mathfrak{m} -full ideal of $R = K[x_1, \dots, x_n]$. Then the equality $\mathfrak{m}I : z = I$ holds for a general linear form z of R . Moreover it is easy to see that, for any $z \in R$, if $\mathfrak{m}I : z = I$, then it implies that $I : \mathfrak{m} = I : z$. Let y_1, \dots, y_s be homogeneous elements in $I : \mathfrak{m}$ such that $\{\overline{y_1}, \dots, \overline{y_s}\}$ is a minimal generating set of $(I : \mathfrak{m})/I$, where $\overline{y_i}$ is the image of y_i in R/I . Then Proposition 2.2 in [3] implies that $\{zy_1, \dots, zy_s\}$ is a part of a minimal generating set of I .

We will prove Theorem 1.1 after a series of lemmas.

Lemma 5.2. *With the same notation as Remark 5.1, write a minimal generating set of I as*

$$zy_1, \dots, zy_s, w_1, \dots, w_t.$$

Let $\overline{w_i}$ be the image of w_i in R/zR and \overline{I} the image of I in R/zR . Then we have:

- (1) $\{\overline{w_1}, \dots, \overline{w_t}\}$ is a minimal generating set of \overline{I} .
- (2) $\mu(I) = \mu(\overline{I}) + l((I : \mathfrak{m})/I)$.

Proof. (1) Suppose that $w_1 \in (w_2, \dots, w_t, z)$. Then

$$(3) \quad w_1 = f_2 w_2 + \dots + f_t w_t + f_{t+1} z$$

for some $f_i \in R$. Since $f_{t+1} z = w_1 - (f_2 w_2 + \dots + f_t w_t) \in I$, we have

$$f_{t+1} \in I : z = I : \mathfrak{m} = (y_1, \dots, y_s, w_1, \dots, w_t).$$

Therefore

$$f_{t+1} = g_1 y_1 + \dots + g_s y_s + h_1 w_1 + \dots + h_t w_t$$

for some $g_i, h_j \in R$, and hence

$$(4) \quad zh_1 w_1 = zf_{t+1} - z(g_1 y_1 + \dots + g_s y_s) - z(h_2 w_2 + \dots + h_t w_t).$$

Thus, from the equalities (3) and (4) above, we obtain

$$w_1 - zh_1 w_1 = (f_2 + zh_2)w_2 + \dots + (f_t + zh_t)w_t + g_1 zy_1 + \dots + g_s zy_s,$$

and $w_1 - zh_1 w_1 \in (zy_1, \dots, zy_s, w_2, \dots, w_t)$. Hence $w_1 \in (zy_1, \dots, zy_s, w_2, \dots, w_t)$, since $\deg(w_1) < \deg(zh_1 w_1)$. This is a contradiction.

(2) immediately follows from (1), since $t = \mu(\overline{I})$ and $s = l((I : \mathfrak{m})/I)$. \square

Lemma 5.3. *Let I be a monomial ideal of $R = K[x_1, \dots, x_n]$. Then I is stable if and only if $(I; x_n, x_{n-1}, \dots, x_1)$ has the complete \mathfrak{m} -full property.*

Proof. The “if” part follows from Example 17 in [6]. So we show the “only if” part. Let u_1, \dots, u_s be monomials in $I : \mathfrak{m}$ such that $\{\overline{u_1}, \dots, \overline{u_s}\}$ is a minimal generating set of $(I : \mathfrak{m})/I$, where $\overline{u_i}$ is the image of u_i in R/I . Then it follows

from Remark 5.1 that $\{x_n u_1, \dots, x_n u_s\}$ is a part of a minimal generating set of I . Write a minimal generating set of I as

$$\mathfrak{B} = \{x_n u_1, \dots, x_n u_s, v_1, \dots, v_t\}$$

where v_1, \dots, v_t are also monomials of I . This is the unique minimal set of monomial generators of I . Hence, to verify that I is stable, it suffices to show that, for each $w \in \mathfrak{B}$, $x_i w / x_{m(w)} \in I$ for every $i < m(w)$. Since $u_j \in I : \mathfrak{m}$, it follows that $x_i(x_n u_j) / x_n = x_i u_j \in I$. Furthermore it follows from Lemma 5.2 (1) that $\{\overline{v_1}, \dots, \overline{v_t}\}$ is a minimal generating set of \overline{I} in $R/x_n R$, and hence x_n does not divide v_j for all j . Therefore, by an inductive argument on the number of variables, we have that $x_i v_j / x_{m(v_j)} \in I$ for every $i < m(v_j)$. \square

Lemma 5.4. *Let I be a completely \mathfrak{m} -full ideal of the polynomial ring $R = K[x_1, \dots, x_n]$. Then $(\text{Gin}(I); x_n, x_{n-1}, \dots, x_1)$ has the completely \mathfrak{m} -full property.*

Proof. Let $B(I)$ and $B(\text{Gin}(I))$ be the sums of the t -sequences of I and $\text{Gin}(I)$ respectively as in Theorem 4.1. Since the t -sequence of I coincides with that of $\text{Gin}(I)$ by Remark 3.5, we see that $B(I) = B(\text{Gin}(I))$. Furthermore the equality $\mu(I) = B(I)$ holds by Theorem 4.1 and the inequality $\mu(\text{Gin}(I)) \leq B(\text{Gin}(I))$ holds by Lemma 4.4. Hence we have

$$B(\text{Gin}(I)) = B(I) = \mu(I) \leq \mu(\text{Gin}(I)) \leq B(\text{Gin}(I)).$$

Therefore the equality $\mu(\text{Gin}(I)) = B(\text{Gin}(I))$ holds. Thus $\text{Gin}(I)$ is completely \mathfrak{m} -full by Theorem 4.1. \square

Lemma 5.5. *Let I be an \mathfrak{m} -full ideal of R , and assume that $\text{Gin}(I)$ is \mathfrak{m} -full. Then*

$$l((I : \mathfrak{m})/I) = l((\text{Gin}(I) : \mathfrak{m})/\text{Gin}(I)).$$

Proof. It suffices to show that

$$H((I : \mathfrak{m})/I, j) = H((\text{Gin}(I) : \mathfrak{m})/\text{Gin}(I), j)$$

for all j . Set $J = \text{Gin}(I)$. Since I and J are \mathfrak{m} -full, there exists a general linear form z of R satisfying $\mathfrak{m}I : z = I$ and $\mathfrak{m}J : z = J$. Then it is easy to see that $I : \mathfrak{m} = I : z$ and $J : \mathfrak{m} = J : z$. Hence, from the exact sequence

$$0 \rightarrow (I : \mathfrak{m})/I \rightarrow R/I \xrightarrow{\times z} R/I \rightarrow R/(I + zR) \rightarrow 0,$$

we have

$$H((I : \mathfrak{m})/I, j-1) = H(R/I + zR, j) - H(R/I, j) + H(R/I, j-1)$$

for all j . Similarly we have

$$H((J : \mathfrak{m})/J, j-1) = H(R/J + zR, j) - H(R/J, j) + H(R/J, j-1)$$

for all j . Recall the well-known facts:

- $H(R/I, j) = H(R/J, j)$ for all j .

- $H(R/(I + zR), j) = H(R/(J + zR), j)$ for all j ([1, Lemma 1.2]).

Hence we get the desired equalities. \square

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. As mentioned in Introduction, it suffices to show that if I is completely \mathfrak{m} -full, then $\text{Gin}(I)$ is stable and $\mu(I) = \mu(\text{Gin}(I))$.

First note that $\text{Gin}(I)$ is stable. This follows from Lemmas 5.3 and 5.4. Next we show that $\mu(I) = \mu(\text{Gin}(I))$. After a generic linear change of variables we may assume that $(I; x_n, x_{n-1}, \dots, x_1)$ has the complete \mathfrak{m} -full property. Since I and $\text{Gin}(I)$ are \mathfrak{m} -full, it follows by Lemma 5.2 (2) that

$$\mu(I) = \mu(\bar{I}) + l((I : \mathfrak{m})/I) \quad \text{and} \quad \mu(J) = \mu(\bar{J}) + l((J : \mathfrak{m})/J).$$

Since \bar{J} is the generic initial ideal of \bar{I} ([4, Corollary 2.15]), it follows by an inductive argument on the number of variables that $\mu(\bar{I}) = \mu(\bar{J})$. Hence the equality $\mu(I) = \mu(J)$ holds by Lemma 5.5. \square

Remark 5.6. If K is a finite field, an ideal can be componentwise linear without being completely \mathfrak{m} -full. To construct an example, suppose that $R = K[x_1, \dots, x_n]$ is the polynomial ring over a finite field K . Assume $n \geq 2$. Let f be the product of *all* linear forms in R and let I be the ideal generated by f and $(x_1, \dots, x_n)^{d+1}$, where $d = \deg f$. Then it is easy to see that the ideal I is componentwise linear but not \mathfrak{m} -full. On the other hand if an ideal $I \subset R$ is completely \mathfrak{m} -full, then I is necessarily componentwise linear. To see this let K' be an infinite field containing K and $R' = R \otimes_K K'$. If $I \subset R$ is completely \mathfrak{m} -full, then the ideal $I' = I \otimes_K K'$ is completely \mathfrak{m} -full in R' . By Theorem 1.1 I' is componentwise linear. This implies that I is componentwise linear, since a minimal free resolution of $I_{<j>}$ over R for any j induces a minimal free resolution of $(I')_{<j>}$ over R' .

Remark 5.7. The original definition of \mathfrak{m} -fullness was suggested to the second author by Rees himself (see Introduction in [8]): An ideal \mathfrak{a} of a local ring (R, \mathfrak{m}) is called \mathfrak{m} -full if $\mathfrak{a}\mathfrak{m} : y = \mathfrak{a}$ for some y in a certain faithfully flat extension of R . If we use this definition, Theorem 1.1 is true without assuming K to be infinite.

6. COMPONENTWISE LINEAR IDEALS OF LOW TYPE

In this section we give a generalization of a theorem of Nagel and Römer, which states that a componentwise linear Gorenstein ideal exists only in embedding dimension one. The following is a consequence of Theorem 1.1.

Proposition 6.1. *Suppose that $R = K[x_1, \dots, x_n]$ is the polynomial ring and I is a componentwise linear ideal of height h such that R/I is Cohen-Macaulay. If the type of I is r and $h \geq r$, then I contains $h - r$ linearly independent linear forms.*

Proof. First note that I is completely \mathfrak{m} -full by Theorem 1.1. Since R/I is Cohen-Macaulay of dimension $n - h$, there exists a regular sequence mod I consisting of $n - h$ linear forms in R , says $\{y_1, \dots, y_{n-h}\}$. Let \bar{I} be the image of I in $\bar{R} = R/(y_1, \dots, y_{n-h})R$. Then \bar{I} is also completely \mathfrak{m} -full in \bar{R} by Corollary 4.5, and hence the equality $\bar{\mathfrak{m}}\bar{I} : \bar{z} = \bar{I}$ holds for some linear form z in R . Therefore it follows that $\bar{I} : \bar{\mathfrak{m}} = \bar{I} : \bar{z}$ (see Remark 5.1), and

$$\begin{aligned} r &= l((I : \mathfrak{m})/I) = l((\bar{I} : \bar{\mathfrak{m}})/\bar{I}) = l((\bar{I} : \bar{z})/\bar{I}) \\ &= l(\bar{R}/(\bar{I} + \bar{z}\bar{R})) = l(R/(I + (z, y_1, \dots, y_{n-h})R)). \end{aligned}$$

Since $l(R/(I + (z, y_1, \dots, y_{n-h})R)) \leq h$ by assumption, it follows that I must contain a regular sequence consisting of $h - r$ linearly independent linear forms. Those linear forms are members of a minimal generating set of I . \square

Corollary 6.2 (Nagel-Römer). *Suppose that $R = K[x_1, \dots, x_n]$ is the polynomial ring and I is a Gorenstein ideal of height h . Then I is componentwise linear if and only if I is a complete intersection ideal minimally generated by at least $h - 1$ linear forms.*

Proof. The “if” part follows from Proposition 6.1. The “only if” part: By Corollary 4.5 it suffices to prove it in the case where I is \mathfrak{m} -primary. It is obvious that I is Gorenstein. By assumption, there exist n linear forms z_1, \dots, z_{n-1}, z_n and an integer $d > 0$ such that $I = (z_1, \dots, z_{n-1}, z_n^d)$. Let \bar{I} be the image of I in R/z_nR . Then the equality $\mu(I) - \mu(\bar{I}) = l(R/I + z_nR)$ holds because $\mu(I) = n$, $\mu(\bar{I}) = n - 1$ and $l(R/I + z_nR) = 1$. Hence I is \mathfrak{m} -full by Lemma 4.3. Furthermore it is obvious that $\bar{I} = (\bar{z}_1, \dots, \bar{z}_{n-1})$ is completely \mathfrak{m} -full in R/z_nR . Therefore I is completely \mathfrak{m} -full, and hence I is componentwise linear by Theorem 1.1. \square

This was proved by Nagel and Römer in Theorem 3.1 of [7]. Our proof as a corollary of Theorem 1.1 and Proposition 6.1 is completely different from theirs. There are also similar results in [2, Theorem 1.1] and [3, Proposition 2.4].

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